A study of negative index materials using transformation optics with applications to super lenses, cloaking, and illusion optics: the scalar case

Hoai-Minh Nguyen *

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Abstract

This paper is devoted to the study of the behavior of the solution $u_\delta \in H^1_0(\Omega)$, as δ goes to 0, to the equation

$$\operatorname{div}(\varepsilon_\delta A \nabla u) + k^2 \varepsilon_0 \Sigma u = \varepsilon_0 f \text{ in } \Omega,$$

where Ω is a smooth open subset of $\mathbb{R}^d$ with $d = 2$ or 3, $f \in L^2(\Omega)$, k is a non-negative constant, A is a uniformly elliptic matrix function, Σ is a real function bounded above and below by positive constants, and ε_δ is a complex function whose the real part takes the value 1 and -1, and the imaginary part is positive and converges to 0 as δ goes to 0. Under some additional general assumptions on A and Σ, we characterize conditions on f under which $\|u_\delta\|_{H^1(\Omega)}$ remains bounded as δ goes to 0. Under these conditions, we also show that u_δ converges weakly in $H^1(\Omega)$ to a limit which is the solution to the limit equation; moreover, we obtain a formula for computing the limit. The applications of these results for perfect lens, cloaking, and illusion optics using negative index materials will be given.

1 Introduction and the statements of the results

1.1 Introduction and the geometric setting

Negative index materials were first investigated theoretically by Veselago in [27] and were innovated by Pendry in [20]. The the existence of such materials was confirmed by Shelby, Smith, and Schultz in [25] (see also [26]). In [20], Pendry showed that the negative index material slab (considered in Veselago's paper) acts as a lens not only for the propagating waves (for which the ray analysis of Veselago is valid) but also for the evanescent near-field radiation. In an earlier work [17], Nicorovici, McPhedran, and Milton showed that negative index materials can be used for a (magnified) cylindrical perfect lens in $2d$ quasi-static regime. In [21], Pendry showed how a cylinder of material can be

*School of Mathematics, University of Minnesota, 206 Church St. SE Minneapolis, MN 55455 USA, hmnguyen@math.umn.edu

designed to magnify an image with the same perfection through conformal transformations from the setting in [20]. In [24], Anathana Ramakrishma and Pendry studied perfect lens solutions to spherical shells for spherical shells of negative index materials. In [11], Lai et. al. investigate the way to create illusion optics by negative index materials again. In [10], Lai et. al. studied cloaking using negative index materials. More applications related to negative index materials can be found in references mentioned in the above works.

In this paper, we extend several results in the works mentioned above to a more general setting for the wave equation in the time harmonic regime. More precisely, we study the behavior of the solution $u_\delta \in H^1_0(\Omega)$, as δ goes to 0, to the equation

$$\operatorname{div}(\varepsilon_\delta A \nabla u) + k^2 \varepsilon_0 \Sigma u = \varepsilon_0 f \text{ in } \Omega,$$

where Ω is a smooth open subset of $\mathbb{R}^d$ with $d = 2$ or 3, $f \in L^2(\Omega)$, k is a non-negative constant, A is a uniformly elliptic matrix function, Σ is a real function bounded above and below by positive constants, and ε_δ is a complex function whose the real part takes the value 1 and -1, and the imaginary part is positive and converges to 0 as δ goes to 0. Under some additional general assumptions on A and Σ, which are very related to the concept of complementary media introduced in [23] (see also [10]), we characterize conditions on $f \in L^2(\Omega)$ under which $\|u_\delta\|_{H^1(\Omega)}$ remains bounded as δ goes to 0. Under these conditions, we also show that u_δ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to a limit which is the solution to the limit equation; moreover, we obtain a formula for computing the limit. The applications of these results for perfect lens, cloaking, and illusion optics using negative index materials will be given. Our results are valid in the quasi-static regime (Theorem 1) as well as in the finite frequency regime (Theorems 2 and A1) for the $2d$ and $3d$ cases. Our approach is based on transformation optics and the calculus of variations (or compactness arguments). Concerning transformation optics, we not only use the fact that the wave equation is invariant under changes of variables but also its Neumann boundary condition (see Proposition 1) [1]. Our approach (see the proof of Theorems 1 and 2 in Sections 2.2 and 3.2) reflects the fact that negative index materials would make the geometry folded as mentioned e.g. in [14]. Our analysis could be generalized to the vectorial (Maxwell) case which would be considered elsewhere. Recently, in [2], the author characterize sources for which cloaking due to anomalous localized resonance appears in the $2d$ quasi-static case with $A = I$. The context in this paper and the one in [2] coincide only in the radial symmetric case. In that case, we emphasize that there is a difference in our characterization and the characterization given there. In this paper, we characterize sources for which the total energy of the fields remains bounded as the loss parameter goes to zero. In [2], the authors characterize sources for which the dissipation energy blows up and the fields remains bounded outside a bounded region as the loss parameter goes to zero. We note that the dissipation energy goes to infinity implies the total energy goes to infinity; however the dissipation energy might be finite even though the total energy blows up (see [13] and Corollary 1).

[1] In some specific situations, the normal derivative on the boundary changes its sign after a change of variables (see (1.25)). This fact will play an important role in our analysis.

Let us describe the problem more precisely. Let Ω, Ω_1, Ω_2, Ω_3, and Ω_4 be smooth open subsets of $\mathbb{R}^d$ $(d = 2, 3)$ such that they are connected [2]

$$\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega_4, \quad \Omega_3 \subset\subset \Omega, \quad \text{and} \quad \Omega \text{ is bounded.}$$

We assume that there exist $x_1 \in \Omega_1$ and $x_2 \in \Omega_3$, and two diffeomorphisms

$$F : \Omega_2 \setminus \{x_1\} \to \Omega_4 \setminus \Omega_2, \quad G : \Omega_4 \setminus \Omega_3 \to \Omega_3 \setminus \{x_2\}$$

such that there exist diffeomorphism extensions of F and G on a neighborhood of $\partial\Omega_2$ and of $\partial\Omega_3$ respectively,

$$\left| \begin{array}{c} F(\Omega_2 \setminus \Omega_1) = \Omega_3 \setminus \bar{\Omega}_2, \\ F(x) = x \text{ on } \partial\Omega_2 \quad \text{and} \quad G(x) = x \text{ on } \partial\Omega_3, \end{array} \right.$$

and

$$G \circ F : \Omega_1 \to \Omega_3 \text{ is a diffeomorphism if one sets } G \circ F(x_1) = x_2. \tag{1.1}$$

The geometry of the problem is given in Figure 1.

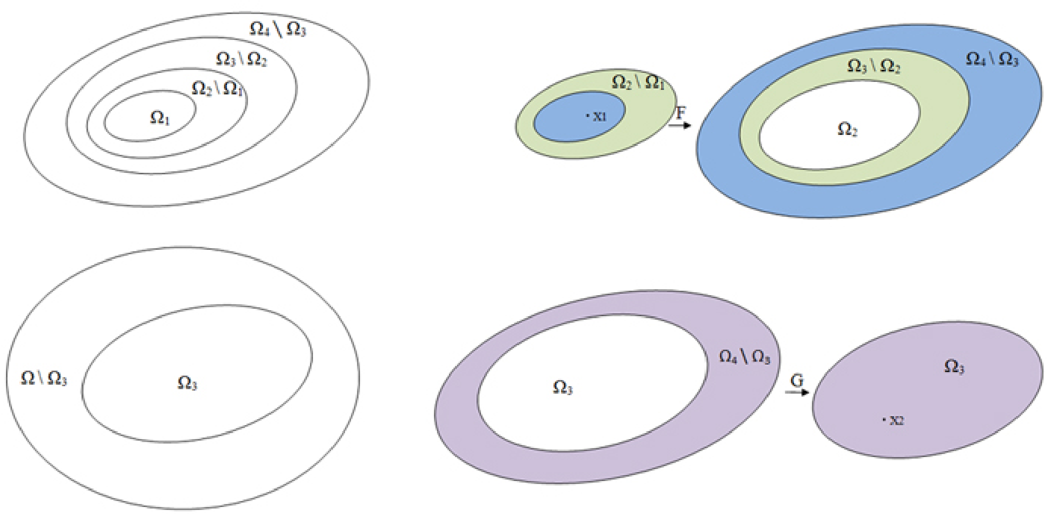

Figure 1: Geometry of the setting

In what follows, B_r denotes the ball centered at 0 with radius r. Here are some examples on the geometry. The first one is

Example 1. *Let* $m, l > 1$, $0 < r_1 < r_2 < r_3 < r$ *and let* $\Omega = B_r$, $\Omega_1 = B_{r_1}$, $\Omega_2 = B_{r_2}$, $\Omega_3 = B_{r_3}$, $\Omega_4 = \mathbb{R}^d$, *and let* $y_0 \in \Omega_3$. *Define*

$$F(x) = \frac{r_2^m}{|x|^m} x \quad \textit{and} \quad G(x) = \frac{r_3^l}{|x|^l} x.$$

[2] In this paper, for two subsets D_1 and D_2 of $\mathbb{R}^d$, $D_1 \subset\subset D_2$ means $\bar{D}_1 \subset D_2$, where $\bar{D}_1$ is the closure of D_1

Assume that $r_3 = r_2^m / r_1^{m-1}$. Then all conditions are verified for $x_1 = x_2 = 0$.

We next give the second example in which Ω_4 is bounded.

Example 2. *Let $0 < r_1 < r_2 < r_3 < r_4$, r and let $\Omega = B_r$, $\Omega_1 = B_{r_1}$, $\Omega_2 = B_{r_2}$, $\Omega_3 = B_{r_3}$, $\Omega_4 = B_{r_4}$. Define*

$$F(x) = \Big(\frac{(r_2 - r_4) r_1 - (r_3 - r_4) r_2}{r_1 r_2^2 - r_1^2 r_2} |x|^2 - \frac{(r_2 - r_4) r_1^2 - (r_3 - r_4) r_2^2}{r_1 r_2^2 - r_1^2 r_2} |x| + r_4 \Big) \frac{x}{|x|},$$

and

$$G(x) = \Big(- \frac{r_3}{r_4 - r_3} |x| + \frac{r_3 r_4}{r_4 - r_3} \Big) \frac{x}{|x|}.$$

All conditions are verified for $x_1 = x_2 = 0$.

For $\delta \ge 0$, ε_δ is defined as follows:

$$\varepsilon_\delta(x) := \begin{cases} -1 + i\delta & \text{if } \Omega_2 \setminus \Omega_1, \\ 1 & \text{otherwise.} \end{cases} \tag{1.2}$$

Let A be a measurable matrix function and Σ be a measurable real function defined on Ω such that

$$\frac{1}{\Lambda} |\xi|^2 \le \langle A(x)\xi, \xi \rangle \le \Lambda |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^d, \tag{1.3}$$

for a.e. $x \in \Omega$, for some $0 < \Lambda < +\infty$, and

$$0 < \operatorname*{ess\,inf}_{\Omega} \Sigma \le \operatorname*{ess\,\mathit{sup}}_{\Omega} \Sigma < +\infty. \tag{1.4}$$

We are interested in the behavior of the solution $u_\delta \in H^1(\Omega)$ to the system

$$\begin{cases} \operatorname{div}(\varepsilon_\delta A \nabla u_\delta) + k^2 \varepsilon_0 \Sigma u_\delta = \varepsilon_0 f & \text{in } \Omega \\ u_\delta = 0 & \text{on } \partial\Omega, \end{cases}$$

as δ goes to 0 [3]. In this equation, the loss parameter involves only in the divergence term. The general case where the loss is also contained in the zero order term will be (briefly) discussed in Sections 1.3.

Our analysis in the zero frequency regime ($k = 0$) and in the finite frequency regime ($k > 0$) share the same approach although technicality is different. For the clarity, we state and prove the results separately. Before presenting these results in the next two subsections, let us introduce the following definitions:

$$\hat{A} := \begin{cases} A & \text{if } x \in \Omega \setminus \Omega_3, \\ G_* F_* A & \text{if } x \in \Omega_3, \end{cases} \qquad \hat{\Sigma} := \begin{cases} \Sigma & \text{if } x \in \Omega \setminus \Omega_3, \\ G_* F_* \Sigma & \text{if } x \in \Omega_3, \end{cases} \tag{1.5}$$

[3] We only consider the Dirichlet boundary condition. Other standard types of boundary conditions could be analyzed similarly.

and

$$\hat{f} := \begin{cases} f & \text{if } x \in \Omega \setminus \Omega_3, \\ G_* F_* f & \text{if } x \in \Omega_3. \end{cases} \tag{1.6}$$

Here and in what follows, we use the standard notations:

$$\mathcal{F}_*\mathcal{A}(y) = \frac{D\mathcal{F}(x)\mathcal{A}(x)D\mathcal{F}^T(x)}{J(x)}, \quad \mathcal{F}_*\Sigma(y) = \frac{\Sigma(x)}{J(x)}, \quad \text{and} \quad \mathcal{F}_*\mathrm{f}(y) = \frac{\mathrm{f}(x)}{J(x)}, \tag{1.7}$$

where $x = \mathcal{F}^{-1}(y)$ and $J(x) = |\det D\mathcal{F}(x)|$.

1.2 Statements of the results in the quasi-static regime

In this section, we are interested in the behavior of u_δ, the unique solution in $H^1_0(\Omega)$ to the equation

$$\operatorname{div}(\varepsilon_\delta A \nabla u_\delta) = \varepsilon_0 f \text{ in } \Omega, \tag{1.8}$$

as δ is a small positive number.

Remark 1. *The existence and uniqueness of u_δ will be established in Section 2.1. An a priori estimate of u_δ will be also given there.*

Here is the main result of this section.

Theorem 1. *Let $\delta > 0$, $f \in L^2(\Omega)$ and $u_\delta \in H^1_0(\Omega)$ be the unique solution to the equation (1.8). Assume $A = F_*A$ in $\Omega_3 \setminus \Omega_2$ and in the case $d = 3$ assume in addition that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$. We have*

a) Case 1: f is compatible with the system (see Definition 1 below). Then the sequence (u_δ) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to u_0 the unique solution in $H^1_0(\Omega)$ to the equation

$$\operatorname{div}(\varepsilon_0 A \nabla u_0) = \varepsilon_0 f \text{ in } \Omega, \tag{1.9}$$

as $\delta \to 0$. Moreover $u_0 = NI(f)$ where $NI(f)$ is given by (1.13)

b) Case 2: f is not compatible with the system. We have

$$\lim_{\delta \to 0} \|u_\delta\|_{H^1(\Omega)} = +\infty. \tag{1.10}$$

In the statement of Theorem 1, we use the following

Definition 1. *Assume $A = F_*A$ in $\Omega_3 \setminus \Omega_2$ and in the case $d = 3$ assume in addition that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$. We say that $f \in L^2(\Omega)$ is compatible with the system if there exist $U \in H^1(\Omega_3 \setminus \Omega_2)$ and $V \in H^1(\Omega_3 \setminus \Omega_2)$ such that*

$$\begin{cases} \operatorname{div}(A\nabla U) = F_*f - f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ U = 0 & \text{on } \partial\Omega_2, \\ A\nabla U \cdot \eta = 0 & \text{on } \partial\Omega_2, \end{cases} \quad \text{and} \quad \begin{cases} \operatorname{div}(A\nabla V) = f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ V = W\Big|_{\text{ext}} & \text{on } \partial\Omega_3, \\ A\nabla V \cdot \eta = A\nabla W \cdot \eta\Big|_{\text{ext}} & \text{on } \partial\Omega_3. \end{cases} \tag{1.11}$$

Here $W \in H^1(\Omega \setminus \partial\Omega_3)$ is the unique solution to the system

$$\begin{cases} \operatorname{div}(\hat{A}\nabla W) = \hat{f} & \text{in } \Omega \setminus \partial\Omega_3 \,, \\ W = 0 & \text{on } \partial\Omega \,, \\ [W] = -U & \text{on } \partial\Omega_3 \,, \\ [\hat{A}\nabla W \cdot \eta] = -A\nabla U \cdot \eta & \text{on } \partial\Omega_3 \,, \end{cases} \tag{1.12}$$

If f is compatible with the system then $NI(f)$ is defined as follows

$$NI(f) = \begin{cases} W & \text{if } x \in \Omega \setminus \Omega_3, \\ V & \text{if } x \in \Omega_3 \setminus \Omega_2, \\ (U+V) \circ F & \text{if } x \in \Omega_2 \setminus \Omega_1, \\ W \circ G \circ F & \text{if } x \in \Omega_1. \end{cases} \tag{1.13}$$

We recall here that $\hat{A}$ and $\hat{f}$ are given in (1.5) and (1.6). In this paper, $[\varphi] := \varphi_{\text{ext}} - \varphi_{\text{int}}$ on ∂D, and η denotes an outward normal unit vector on ∂D for any smooth open bounded subset D of $\mathbb{R}^d$.

Remark 2. *The uniqueness of U and V follow from the unique continuation principle (see e.g. [22] and [1]) and the fact that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$ in the 3d case. The existence and uniqueness of W are standard and can be proved by Fredholm's theory. The details are left to the reader.*

Remark 3. *In general, $\|u_0\|_{H^1(\Omega)}$ is not* uniformly *bounded with respect to $\|f\|_{L^2}$ i.e., there does not exist a positive constant C, depending only on the geometry, Σ, an A, such that*

$$\|u_0\|_{H^1} \le C\|f\|_{L^2}.$$

This is a consequence of the fact that $\|u_\delta\|_{H^1}$ is bounded only for the class of compatible functions $f \in L^2(\Omega)$ which is not open in $L^2(\Omega)$ in general. Hence there is a difference on the stability between equation (1.8) with $\delta = 0$ and standard elliptic equations of divergence form.

Remark 4. *We note that since $F_*A = A$ in $\Omega_3 \setminus \Omega_2$, $U + V$ $(\in H^1(\Omega_3 \setminus \bar{\Omega}_2)$ satisfies:*

$$\begin{cases} \operatorname{div}\Big(F_*A\nabla(U+V)\Big) = F_*f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ U + V = V & \text{on } \partial\Omega_2, \\ F_*A\nabla(U+V)\cdot\eta = F_*A\nabla V \cdot \eta & \text{on } \partial\Omega_2. \end{cases}$$

From Theorem 1, it is clear that if f is compatible and $f = 0$ in Ω_3, then $u_0 = u_0 \circ G \circ F$ on $\partial\Omega_1$ since $U = 0$. This is a phenomena of perfect lenses. More application of Theorem 1 on perfect lenses, cloaking, and illusion optics will be given in Section 4. The fact that in the context of Theorem 1, there are sources such that the energy of the fields goes to

infinity has been observed in several situations see e.g. [13], [6]. The media $-A$ in Ω_2 and F_*A in $\Omega_3 \setminus \Omega_2$ might be called complementary medium as suggested in [21] (see also [10] and Section 4).

Assume $f = 0$ in Ω_3. From the definition of U (1.11) and the unique continuation principle, it follows that $U = 0$ in $\Omega_3 \setminus \bar{\Omega}_2$. Therefore, by (1.12), $W \in H^1_0(\Omega)$ is the solution to the equation

$$\operatorname{div}(\hat{A}\nabla W) = \hat{f} \text{ in } \Omega.$$

Assume in addition that $\hat{A} = A$ in $\Omega_3 \setminus \Omega_2$ i.e. $G_*F_*A = A$ in $\Omega_3 \setminus \Omega_2$. It follows from the definition of V (1.11) and the unique continuation principle that $V = W$ in $\Omega_3 \setminus \Omega_2$. Hence we obtain the following corollary

Corollary 1. *Let $d = 2, 3$ and $f \in L^2(\Omega)$ be such that $f = 0$ in Ω_3. Assume that $G_*F_*A = F_*A = A$ in $\Omega_3 \setminus \bar{\Omega}_2$, and in the case $d = 3$ assume in addition that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$. Then f is compatible and $(u_\delta) \subset H^1_0(\Omega)$, the solutions to the equation (1.8), converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to u_0 the unique solution in $H^1_0(\Omega)$ to the equation*

$$\operatorname{div}(\varepsilon_0 A \nabla u_0) = \varepsilon_0 f \text{ in } \Omega,$$

as $\delta \to 0$. Moreover,

$$u_0 = W \text{ in } \Omega \setminus \Omega_3,$$

where $W \in H^1_0(\Omega)$ is the unique solution to the equation

$$\operatorname{div}(\hat{A}\nabla W) = f \text{ in } \Omega.$$

We recall here that $\hat{A}$ is defined in (1.5).

1.3 Statements of the results in the finite frequency regime

In this section, we consider the finite frequency regime. We are interested in the behavior of u_δ, the unique solution in $H^1_0(\Omega)$ to the equation

$$\operatorname{div}(\varepsilon_\delta A \nabla u_\delta) + k^2 \varepsilon_0 \Sigma u_\delta = \varepsilon_0 f \text{ in } \Omega, \tag{1.14}$$

as δ is a small positive number. We assume in addition that the following systems

$$\begin{cases} \operatorname{div}(A\nabla v) + k^2 \Sigma v = 0 & \text{in } \Omega \setminus \bar{\Omega}_2 , \\ v = 0 & \text{on } \partial\Omega , \\ A\nabla v \cdot \eta = 0 & \text{on } \partial\Omega_2 , \end{cases} \tag{1.15}$$

and

$$\begin{cases} \operatorname{div}(\hat{A}\nabla U) + k^2 \hat{\Sigma} U = 0 & \text{in } \Omega , \\ U = 0 & \text{on } \partial\Omega , \end{cases} \tag{1.16}$$

have only zero solution in $H^1(\Omega \setminus \bar{\Omega}_2)$ and $H^1(\Omega)$ respectively [4].

[4] If we consider scattered waves, these conditions are not necessary. However, we need to impose the outgoing condition, see Appendix A.

Remark 5. *The existence and uniqueness of u_δ will be established in Section 3.1. An a priori estimate of u_δ will be also given there.*

The main result in this section is

Theorem 2. *Let $d = 2, 3$, $\delta > 0$, $f \in L^2(\Omega)$ and in the 3d case assume in addition that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$. Let $u_\delta \in H^1_0(\Omega)$ be the unique solution to the equation (1.14). Assume that $F_* A = A$ and $F_* \Sigma = \Sigma$ in $\Omega_3 \setminus \Omega_2$. We have*

a) Case 1: f is compatible with the system. Then the sequence (u_δ) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to $u_0 \in H^1_0(\Omega)$, the unique solution to the equation

$$\operatorname{div}(\varepsilon_0 \nabla u_0) + k^2 \varepsilon_0 \Sigma u_0 = \varepsilon_0 f \text{ in } \Omega. \tag{1.17}$$

as $\delta \to 0$. Moreover, $u_0 = NL(k, f)$ where $NL(k, f)$ is given in (1.22)

b) Case 2: f is not compatible with the system. We have

$$\lim_{\delta \to 0} \|u_\delta\|_{H^1(\Omega)} = +\infty. \tag{1.18}$$

In Theorem 2 we use the following

Definition 2. *We say that $f \in L^2(\Omega)$ is compatible with the system if there exist $U \in H^1(\Omega_3 \setminus \Omega_2)$ and $V \in H^1(\Omega_3 \setminus \Omega_2)$ such that*

$$\begin{cases} \operatorname{div}(A\nabla U) + k^2 \Sigma U = F_* f - f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ U = 0 & \text{on } \partial\Omega_2, \\ A\nabla U \cdot \eta = 0 & \text{on } \partial\Omega_2, \end{cases} \tag{1.19}$$

and

$$\begin{cases} \operatorname{div}(A\nabla V) + k^2 \Sigma V = f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ V = W\Big|_{\text{ext}} & \text{on } \partial\Omega_3, \\ A\nabla V \cdot \eta = A\nabla W \cdot \eta\Big|_{\text{ext}} & \text{on } \partial\Omega_3. \end{cases} \tag{1.20}$$

Here $W \in H^1(\Omega \setminus \partial\Omega_3)$ is the unique solution to the system

$$\begin{cases} \operatorname{div}(\hat{A}\nabla W) + k^2 \hat{\Sigma} W = \hat{f} & \text{in } \Omega \setminus \partial\Omega_3 , \\ W = 0 & \text{on } \partial\Omega , \\ [W] = -U & \text{on } \partial\Omega_3 , \\ [\hat{A}\nabla W \cdot \eta] = -A\nabla U \cdot \eta & \text{on } \partial\Omega_3 . \end{cases} \tag{1.21}$$

Then $NI(k, f)$ is defined as follows

$$NI(k, f) = \begin{cases} W & \text{if } x \in \Omega \setminus \Omega_3, \\ V & \text{if } x \in \Omega_3 \setminus \Omega_2, \\ (U + V) \circ F & \text{if } x \in \Omega_2 \setminus \Omega_1, \\ W \circ G \circ F & \text{if } x \in \Omega_1. \end{cases} \tag{1.22}$$

We recall here that $\hat{A}$ and $\hat{f}$ are given in (1.5) and (1.6).

Remark 6. *The uniqueness of U and V follow from the unique continuation principle see e.g. [22] and [1] and the fact that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$ in the 3d case. The existence and uniqueness of W are standard and can be proved by Fredholm's theory since the system* (1.16) *is well-posed.*

Remark 7. *We will present the results on scattered waves in the appendix.*

Remark 8. *In general, $\|u_0\|_{H^1(\Omega)}$ is not uniformly bounded with respect to $\|f\|_{L^2}$ (see Remark 3).*

Remark 9. *We note that since $F_*A = A$ and $F_*\Sigma = \Sigma$ in $\Omega_3 \setminus \Omega_2$, $U + V \in H^1(\Omega_3 \setminus \bar{\Omega}_2)$ and satisfies:*

$$\begin{cases} \operatorname{div}\Big(F_*A\nabla(U+V)\Big) + k^2 F_*\Sigma(U+V) = F_*f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ U + V = V & \text{on } \partial\Omega_2, \\ F_*A\nabla(U+V)\cdot\eta = F_*A\nabla V \cdot \eta & \text{on } \partial\Omega_2. \end{cases}$$

Applications of Theorem 2 for perfect lenses, cloaking, and illusion optics will be given in Section 4. The media $-A, -\Sigma$ in Ω_2 and $F_*A, F_*\Sigma$ in $\Omega_3 \setminus \Omega_2$ might be called complementary medium as suggested in [21] (see also [10] and Section 4).

Assume $f = 0$ in Ω_3. From the definition of U (1.19) and the unique continuation principle, it follows that $U = 0$ in $\Omega_3 \setminus \bar{\Omega}_2$. Therefore, by (1.12), $W \in H^1_0(\Omega)$ is the solution to the equation

$$\operatorname{div}(\hat{A}\nabla W) + k^2\hat{\Sigma}W = f \text{ in } \Omega.$$

Assume in addition that $\hat{A} = A$ and $\hat{\Sigma} = \Sigma$ in $\Omega_3 \setminus \Omega_2$ i.e. $G_*F_*A = A$ and $G_*F_*\Sigma = \Sigma$ in $\Omega_3 \setminus \Omega_2$. It follows from the definition of V (1.11) and the unique continuation principle that $V = W$ in $\Omega_3 \setminus \Omega_2$. Hence we obtain the following corollary

Corollary 2. *Assume $G_*F_*A = F_*A = A$ in $\Omega_3 \setminus \Omega_2$, $f = 0$ in Ω_3, and in the 3d case assume in addition that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$. Then f is compatible and (u_δ), the solution to the equation* (1.14) *in $H^1_0(\Omega)$, converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to u_0 the unique solution in $H^1_0(\Omega)$ to the equation*

$$\operatorname{div}(\varepsilon_0 A\nabla u_0) + k^2\varepsilon_0\Sigma u_0 = \varepsilon_0 f \text{ in } \Omega,$$

as $\delta \to 0$. Moreover

$$u_0 = W \text{ in } \Omega \setminus \Omega_3,$$

where $W \in H^1_0(\Omega)$ is the unique solution to the equation

$$\operatorname{div}(\hat{A}\nabla W) + k^2\hat{\Sigma}W = f \text{ in } \Omega.$$

We recall here that $\hat{A}$ and $\hat{\Sigma}$ are defined in (1.5).

In Theorem 2, we just consider the case in which the loss parameter involves only in the divergence term. In fact, the analysis in this paper can be extended to the general case in which the loss parameter involves in both terms. We have

Theorem 3. *Let $d = 2, 3$, $\delta > 0$, $f \in L^2(\Omega)$ and in the 3d case assume in addition that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$. Let $u_\delta \in H^1_0(\Omega)$ be the unique solution to the equation*

$$\operatorname{div}(\varepsilon_\delta A \nabla u_\delta) + k^2 \varepsilon_\delta \Sigma u_\delta = f.$$

Assume that $F_* A = A$ and $F_* \Sigma = \Sigma$ in $\Omega_3 \setminus \Omega_2$. We have

a) Case 1: f is compatible with the system. Then the sequence (u_δ) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to $u_0 \in H^1_0(\Omega)$, the unique solution to the equation

$$\operatorname{div}(\varepsilon_0 \nabla u_0) + k^2 \varepsilon_0 \Sigma u_0 = \varepsilon_0 f \text{ in } \Omega. \tag{1.23}$$

as $\delta \to 0$. Moreover, $u_0 = NL(k, f)$ where $NL(k, f)$ is given in (1.22)

b) Case 2: f is not compatible with the system. We have

$$\lim_{\delta \to 0} \|u_\delta\|_{H^1(\Omega)} = +\infty. \tag{1.24}$$

The proof of Theorem 3 follows similarly as the one of Theorem 2 (see Section 3). The details are left to the reader.

1.4 Ideas of the proofs of Theorems 1 and 2

In this section we describe briefly the ideas of the proofs of Theorems 1 and 2. The proof of Theorem 1 is proceeded as follows. We first show that if f is compatible then $u \in H^1_0(\Omega)$ is a solution to the equation (1.8) if and only if $u = NI(f)$. This is the crucial step of the proof. The key ingredients for the proof of this fact are properties of transformation optics stated in Proposition 1 below where we essentially use (1.25). Using the fact that $NI(f)$ is the unique solution in $H^1_0(\Omega)$ to the equation (1.8) with $\delta = 0$, we can easily establish the boundedness of u_δ in $H^1(\Omega)$ by applying an elementary a priori estimate for the solution of equations of class (1.8) (Lemma 1) to the difference $u_\delta - u_0$. We emphasize here that the constant in the estimate of Lemma 1 blows up as $1/\delta$ as δ goes to 0 however, the source in the corresponding equation is of the order δ: it is exactly the place where we make use the existence of u_0. The convergence of u_δ follows immediately from its boundedness and the existence and uniqueness of u_0. The proof of Theorem 2 follows in the same way. The only difference between these two proofs lies mainly on the way we establish a priori estimates for the solutions to the corresponding classes ((1.8) vs (1.14)); the proof in the finite frequency case is more complicated (see the proof of Propositions 2 and 3). This situation happens naturally and quite often since obtaining estimates for the finite frequency case usually requires more work than the quasi-static case.

Remark 10. *In this paper, we only consider structures generated from the class of transformations F and G mentioned in Section 1.1. As a matter of fact, negative index material regions discussed in this paper are finite. Nevertheless, our analysis is quite robust and could be generalized for other situations e.g. the ones considered in [20] [19], and [8].*

We recall the following well-known proposition, which plays a very important role in our approach, on the transformation optics or the change of variables formula for the time harmonic wave equation.

Proposition 1. *Let $k \geq 0$, D_1 and D_2 are two open subsets of $\mathbb{R}^d$, T be a diffeomorphism from D_1 onto D_2, $a \in [L^\infty(D_1)]^{d\times d}$ and $\sigma \in L^\infty(D_1)$. Fix $u \in H^1(D_1)$ and set $v = u\circ T^{-1}$. We have*

$$\operatorname{div}(a\nabla u) + k^2\sigma u = f$$

iff

$$\operatorname{div}(T_*a\nabla v) + k^2 T_*\sigma v = T_*f$$

Assume that Γ_1 and Γ_2 are open subsets of ∂D_1 and ∂D_2 such that Γ_1 and Γ_2 are smooth, $\Gamma_2 = T(\Gamma_1)$, and $\mathbf{T} := T\Big|_{\Gamma_1} \Gamma_1 \to \Gamma_2$ is a diffeomorphism. We have

$$a\nabla u \cdot \eta_1 = g_1 \text{ on } \Gamma_1$$

iff

$$T_*a\nabla v \cdot \eta_2 = g_2 \text{ on } \Gamma_2,$$

where [5]

$$g_2(y) = g_1(x)/|\det \nabla \mathbf{T}(x)| \text{ with } x = \mathbf{T}^{-1}(y).$$

Here η_1 and η_2 are normal unit vector on Γ_1 and Γ_2 directed to the exterior of D_1 and D_2. In particular, if $\Gamma_1 = \Gamma_2$, $\mathbf{T}(x) = x$ on Γ_1, $D_2 \cap D_1 = \emptyset$. We have

$$T_*a\nabla v \cdot \eta_1 = -a\nabla u \cdot \eta_1 \text{ on } \Gamma_1 = \Gamma_2. \tag{1.25}$$

Proof. Proposition 1 is a consequence of the change of variables. The first equivalent relation is well-known and can be proved by using the weak formula. The second equivalent follows similarly. A close variance and the approach mentioned here can be found in [9, Section 2.2]. The details are left to the reader. □

Remark 11. *Identity* (1.25) *gives a connection between negative index material and transformation optics. This identity will play an important role in our analysis.*

1.5 The organization of the paper

The paper is organized as follows. In Section 2 and 3, we respectively establish Theorems 1 and 2. Section 4 is devoted to applications of Theorem 1 and 2. The results related to the scattered waves will be presented in the appendix.

[5] In the identity below, $\nabla\mathbf{T}$ stands for the gradient of a transformation from a $(d-1)$-manifold into a $(d-1)$-manifold, and $\det \nabla\mathbf{T}$ denotes the determinant of $(d-1)\times(d-1)$ matrix.

2 Quasi-static regime - Proof of Theorem 1

This section which contains two subsections is devoted to the proof of Theorem 1. In the first subsection, we establish basis properties of the solution to equation (1.8) such as existence, uniqueness, and an a priori estimate. In the second subsection, we give the proof of Theorem of 1.

2.1 Preliminaries

The main result of this section is the following proposition which implies the existence and uniqueness of u_δ, and give an a priori estimate for it.

Proposition 2. *Let $d = 2, 3$, $g \in H^{-1}(\Omega)$ (the duality of $H^1_0(\Omega)$), and $A \in [L^\infty(\Omega)]^{d\times d}$. Assume that A satisfies* (1.3)*. There exists a unique solution $v_\delta \in H^1_0(\Omega)$ to the equation*

$$\operatorname{div}(\varepsilon_\delta A \nabla v_\delta) = g \text{ in } \Omega.$$

Moreover, we have

$$\|v_\delta\|_{H^1(\Omega)} \le C \max\{1, 1/\delta\} \|g\|_{H^{-1}(\Omega)}. \tag{2.1}$$

In this section, C denotes a positive constant depending only on Ω and Λ.

Proof. By Lax-Milgram's theorem, it suffices to verify that

$$\Big| \int_\Omega \varepsilon_\delta \langle A\nabla u, \nabla u\rangle \, dx \Big| \ge C \min\{\delta, 1\} \int_\Omega |\nabla u|^2 \, dx \quad \forall\, u \in H^1_0(\Omega) \tag{2.2}$$

for some positive constant C. Indeed, we have

$$\Big| \int_\Omega \varepsilon_\delta \langle A\nabla u, \nabla u\rangle \, dx \Big|^2 \ge (a-b)^2 + \delta^2 b^2,$$

where

$$a = \int_{\Omega\setminus(\Omega_2\setminus\Omega_1)} \langle A\nabla u, \nabla u\rangle \, dx \quad \text{ and } \quad b = \int_{\Omega_2\setminus\Omega_1} \langle A\nabla u, \nabla u\rangle \, dx.$$

On the other hand, for $\delta < 1/2$

$$\begin{aligned}(a-b)^2 + \delta^2 b^2 =& (1-\delta^2/2)a^2 - 2ab + b^2/(1-\delta^2/2) + \delta^2 a^2/2 + (\delta^2/2 - \delta^4/2)b^2/(1-\delta^2/2)\\ \ge& \Big[(1-\delta^2/2)^{1/2} a - b/(1-\delta^2/2)^{1/2}\Big]^2 + C\delta^2(a^2+b^2),\end{aligned}$$

and for $\delta > 1/2$,

$$(a-b)^2 + \delta^2 b^2 \ge (a-b)^2 + b^2/4 \ge C(a^2+b^2).$$

Therefore, we obtain (2.2). □

2.2 Proof of Theorem 1

Case 1: f is compatible. The proof is derived from the following steps:

Step 1: Let $v \in H^1_0(\Omega)$ be a solution to the equation

$$\operatorname{div}(\varepsilon_0 A \nabla v) = \varepsilon_0 f. \tag{2.3}$$

We prove that $v = NL(f)$.

Step 2: Define $u_0 := NI(f)$. We prove that $u_0 \in H^1_0(\Omega)$ is a solution to the equation

$$\operatorname{div}(\varepsilon_0 A \nabla u_0) = \varepsilon_0 f.$$

Step 3: We prove that $(u_\delta)_{0<\delta<1}$ is bounded in $H^1(\Omega)$.

Step 4: We prove that (u_δ) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to u_0 as δ goes to 0.

We note immediately here that as a consequence of Steps 1 and 2, $NI(f)$ is the unique solution in $H^1_0(\Omega)$ to the equation (1.8). We now process the proof of these steps.

Step 1: Assume $v \in H^1_0(\Omega)$ is a solution to the equation

$$\operatorname{div}(\varepsilon_0 A \nabla v) = \varepsilon_0 f \text{ in } \Omega.$$

Set

$$v_1 = v \circ F^{-1} \text{ in } \Omega_4 \setminus \Omega_2 \tag{2.4}$$

and

$$\hat{\varepsilon}_0 = \begin{cases} 1 & \text{if } x \in \Omega_3 \setminus \Omega_2, \\ -1 & \text{if } x \in \Omega_4 \setminus \Omega_3. \end{cases} \tag{2.5}$$

From Proposition 1, $v_1 \in H^1(\Omega_3 \setminus \bar{\Omega}_2) \cap H^1_{loc}(\Omega_4 \setminus \bar{\Omega}_2)$ and v_1 satisfies

$$\operatorname{div}(\hat{\varepsilon}_0 F_* A \nabla v_1) = \hat{\varepsilon}_0 F_* f \text{ in } \Omega_4 \setminus \bar{\Omega}_2, \tag{2.6}$$

and

$$v_1 = v \text{ on } \partial\Omega_2 \quad \text{ and } \quad F_* A \nabla v_1 \cdot \eta = A \nabla v . \eta \Big|_{\text{ext}} \text{ on } \partial\Omega_2.$$

In the last identity, we use the fact that $F_* A \nabla v_1 \cdot \eta = -A \nabla v \cdot \eta \Big|_{\text{int}}$ on $\partial\Omega_2$ by Proposition 1 and $A \nabla v \cdot \eta \Big|_{\text{ext}} = -A \nabla v \cdot \eta \Big|_{\text{int}}$ on $\partial\Omega_2$ by the transmission condition on $\partial\Omega_2$. Define

$$\mathbf{U} = v_1 - v \text{ in } \Omega_3 \setminus \Omega_2. \tag{2.7}$$

Since $F_* A = A$ in $\Omega_3 \setminus \Omega_2$, it follows that

$$\begin{cases} \operatorname{div}(A \nabla \mathbf{U}) = F_* f - f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ \mathbf{U} = 0 & \text{on } \partial\Omega_2, \\ A \nabla \mathbf{U} \cdot \eta = 0 & \text{on } \partial\Omega_2. \end{cases} \tag{2.8}$$

Since $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$ in the $3d$ case, applying the unique continuation principle, see e.g., [22] and [1], we have

$$\mathbf{U} = U \text{ in } \Omega_3 \setminus \Omega_2. \tag{2.9}$$

We recall here that U is defined in (1.11). Define the function v_2 on Ω as follows

$$v_2(x) = \begin{cases} v_1 \circ G^{-1} & \text{if } x \in \Omega_3, \\ v & \text{if } x \in \Omega \setminus \Omega_3. \end{cases} \tag{2.10}$$

Using (1.1) and applying Proposition 1, we have

$$\operatorname{div}(\hat{A}\nabla v_2) = \hat{f} \text{ in } \Omega \setminus \partial\Omega_3, \tag{2.11}$$

and, on $\partial\Omega_3$,

$$\begin{aligned} \hat{A}\nabla v_2 \cdot \eta\Big|_{\text{ext}} - \hat{A}\nabla v_2 \cdot \eta\Big|_{\text{int}} &= A\nabla v \cdot \eta\Big|_{\text{ext}} + F_* A\nabla v_1 \cdot \eta\Big|_{\text{ext}} \\ &= A\nabla v \cdot \eta\Big|_{\text{ext}} - F_* A\nabla v_1 \cdot \eta\Big|_{\text{int}} \quad \text{(by (2.6))} \\ &= A\nabla v \cdot \eta\Big|_{\text{ext}} - F_* A\nabla (v + \mathbf{U}) \cdot \eta\Big|_{\text{int}} \quad \text{(by (2.7))} \end{aligned}$$

It follows from (2.9) that

$$\hat{A}\nabla v_2 \cdot \eta\Big|_{\text{ext}} - \hat{A}\nabla v_2 \cdot \eta\Big|_{\text{int}} = -F_* A\nabla U \cdot \eta\Big|_{\text{int}}. \tag{2.12}$$

Since $G(x) = x$ on $\partial\Omega$, we also obtain, on $\partial\Omega_3$,

$$v_2\Big|_{\text{ext}} - v_2\Big|_{\text{int}} = v\Big|_{\text{ext}} - v_1\Big|_{\text{ext}} = v\Big|_{\text{ext}} - v_1\Big|_{\text{int}} = v\Big|_{\text{ext}} - (\mathbf{U} + v)\Big|_{\text{int}} = -U \tag{2.13}$$

A combination of (2.11), (2.12), (2.13), and (1.12) yields

$$v_2 = W \text{ in } \Omega. \tag{2.14}$$

This implies that v satisfies

$$\begin{cases} \operatorname{div}(A\nabla v) = f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ v = W\Big|_{\text{ext}} & \text{on } \partial\Omega_3, \\ A\nabla v \cdot \eta\Big|_{\text{int}} = A\nabla W \cdot \eta\Big|_{\text{ext}} & \text{on } \partial\Omega_3. \end{cases}$$

Since $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$ in the $3d$ case, applying the unique continuation principle, see e.g., [22] and [1], we have

$$v = V \text{ in } \Omega_3 \setminus \bar{\Omega}_2. \tag{2.15}$$

From (2.10), (2.7), and (2.4), it follows that

$$v = \begin{cases} v_2 & \text{in } \Omega \setminus \Omega_3, \\ \mathbf{U} + v_1 & \text{in } \Omega_3 \setminus \Omega_2, \\ v_1 \circ F & \text{in } \Omega_2 \setminus \Omega_1, \\ v_2 \circ G \circ F & \text{in } \Omega_1, \end{cases}$$

We deduce from (2.9), (2.14), and (2.15) that

$$v = \begin{cases} W & \text{in } \Omega \setminus \Omega_3, \\ V & \text{in } \Omega_3 \setminus \Omega_2, \\ (V+U) \circ F & \text{in } \Omega_2 \setminus \Omega_1, \\ W \circ G \circ F & \text{in } \Omega_1, \end{cases}$$

Therefore,

$$v = NL(f) \text{ in } \Omega.$$

The proof of Step 1 completes.

Step 2: We first prove that

$$\operatorname{div}(A\nabla u_0) = f \text{ in } (\Omega \setminus \bar{\Omega}_1) \setminus (\partial\Omega_3 \cup \partial\Omega_2). \tag{2.16}$$

where $u_0 := NI(f)$. Since

$$\operatorname{div}(A\nabla W) = f \text{ in } \Omega \setminus \bar{\Omega}_3$$

and

$$\operatorname{div}(A\nabla V) = f \text{ in } \Omega_3 \setminus \bar{\Omega}_2,$$

it follows that

$$\operatorname{div}(A\nabla u_0) = f \text{ in } \Big(\Omega \setminus \bar{\Omega}_3\Big) \cup \Big(\Omega_3 \setminus \bar{\Omega}_2\Big). \tag{2.17}$$

Since $F_*A = A$ in $\Omega_3 \setminus \Omega_2$, $U + V \in H^1(\Omega_3 \setminus \bar{\Omega}_2)$ and satisfies:

$$\operatorname{div}\Big(F_*A\nabla(U+V)\Big) = F_*f \text{ in } \Omega_3 \setminus \bar{\Omega}_2,$$

we obtain

$$\operatorname{div}(A\nabla u_0) = f \text{ in } \Omega_2 \setminus \bar{\Omega}_1. \tag{2.18}$$

A combination of (2.17) and (2.18) yields (2.16).

It remains to verify

$$[A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_3, \quad [\varepsilon_0 A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_2 \tag{2.19}$$

and

$$\operatorname{div}(\varepsilon_0 A\nabla u_0) = \varepsilon_0 f \text{ in } \Omega_2. \tag{2.20}$$

From the definition of V (1.11), of W (1.12), and of $NI(f)$ (1.13), we have

$$[A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_3.$$

Since $U = 0$ and $A\nabla U \cdot \eta = 0$ on $\partial\Omega_2$, it follows from Proposition 1 that

$$[\varepsilon_0 A\nabla u_0] = [u_0] = 0 \text{ on } \partial\Omega_2,$$

Define

$$V_1 = \begin{cases} W \circ G & \text{in } \Omega_4 \setminus \Omega_3, \\ V + U & \text{in } \Omega_3 \setminus \Omega_2. \end{cases}$$

By Proposition 1, we have

$$\operatorname{div}(\hat{\varepsilon}_0 F_* A\nabla V_1) = \hat{\varepsilon}_0 F_* f \text{ in } \Omega_4 \setminus \bar{\Omega}_2,$$

where $\hat{\varepsilon}_0$ is given in (2.5). From Proposition 1, it follows that

$$\operatorname{div}(\varepsilon_0 A\nabla u_0) = \varepsilon_0 f \text{ in } \Omega_2.$$

Step 3: Set

$$v_\delta = u_\delta - u_0 \text{ in } \Omega.$$

We have, in Ω,

$$\operatorname{div}(\varepsilon_\delta A\nabla v_\delta) = \operatorname{div}(\varepsilon_\delta A\nabla u_\delta) - \operatorname{div}(\varepsilon_\delta A\nabla u_0) = \varepsilon_0 f - \operatorname{div}(\varepsilon_0 A\nabla u_0) + \operatorname{div}\big[(\varepsilon_0 - \varepsilon_\delta)A\nabla u_0\big].$$

Since $\operatorname{div}(\varepsilon_0 \nabla u_0) = \varepsilon_0 f$ in Ω, we obtain

$$\operatorname{div}(\varepsilon_\delta \nabla v_\delta) = \operatorname{div}\big[(\varepsilon_0 - \varepsilon_\delta)A\nabla u_0\big] \text{ in } \Omega.$$

We deduce from Proposition 2 that

$$\|\nabla v_\delta\|_{L^2(\Omega)} \le C\|\nabla u_0\|_{L^2(\Omega)},$$

which yields, since $u_\delta = v_\delta + u_0$,

$$\|\nabla u_\delta\|_{L^2(\Omega)} \le C\|\nabla u_0\|_{L^2(\Omega)}.$$

Since $u_\delta \in H^1_0(\Omega)$, by Poincaré's inequality, it follows that

$$\|u_\delta\|_{H^1(\Omega)} \le C\|\nabla u_0\|_{L^2(\Omega)}.$$

Step 3 completes.

Step 4: The conclusion of Step 4 follows from Step 3 and the fact that the limit of u_δ (up to a subsequence) satisfies (2.3) and this equation has a unique solution in $H^1_0(\Omega)$.

Case 2: f is not compatible with the system. We process the proof by contradiction as follows. Assume that (1.10). Without loss of generality, there exists a sequence (u_δ) (up

to an extraction) bounded in $H^1(\Omega)$ such that u_δ converges weakly to u in $H^1(\Omega)$. It follows that $u \in H^1_0(\Omega)$ is a solution to the equation

$$\operatorname{div}(\varepsilon_0 A \nabla u) = \varepsilon_0 f \text{ in } \Omega.$$

Define

$$U = u_0 \circ F^{-1} - u_0 \text{ in } \Omega_3 \setminus \Omega_2 \quad \text{ and } \quad V = u_0 \text{ in } \Omega_3 \setminus \Omega_2.$$

This implies, as in Step 1, U and V satisfy (1.11) respectively. We have a contradiction since f is not compatible with the system. $\square$

Remark 12. *Our analysis (mainly in Steps 1 and 2) would make media whose some region contains negative index materials behave like a folded space as noted in [14].*

3 Finite frequency regime - Proof of Theorem 2

This section which contains two subsections is devoted to the proof of Theorem 2. In the first subsection, we establish basis properties of the solution to equation (1.14) such as existence, uniqueness, and an a priori estimate. In the second subsection, we give the proof of Theorem of 2. In this section, we always assume that (1.15) and (1.16) have only zero solution.

3.1 Preliminaries

In this section, we prove the the existence and uniqueness of the system (1.14) and establish an a priori estimate for its solutions. For this end, we need the following three lemmas. The first one is

Lemma 1. *Let $d \geq 2$, O be a smooth bounded open subset of $\mathbb{R}^d$ and $\Gamma \subset O$ be such that $\mathcal{H}^{d-1}(\Gamma) > 0$[6]. Assume $u \in H^1(O)$. We have*

$$\|u\|_{L^2(O)} \leq C\Big(\|\nabla u\|_{L^2(O)} + \|u\|_{L^2(\Gamma)}\Big),$$

for some positive constant C independent of u.

Proof. The proof follows from standard compactness arguments. The details are left to the reader. $\square$

The second one is from [7, Lemma A.1]:

Lemma 2. *Let $d \geq 2$, O be a smooth bounded open subset of $\mathbb{R}^d$, and $u \in H_{\operatorname{div}}(O)$, i.e., $u \in [L^2(O)]^d$ and $\operatorname{div} u \in L^2(O)$. We have*

$$\|u \cdot n\|^2_{H^{-1/2}(\partial O)} \leq C \|u\|_{L^2(O)}\Big(\|\operatorname{div} u\|_{L^2(O)} + \|u\|_{L^2(O)}\Big),$$

for some positive constant C depending only on O.

[6] $\mathcal{H}^{d-1}$ denotes the $d-1$-Hausdorf measure.

Remark 13. *The standard estimate in the* $H_{\mathrm{div}}(O)$ *is:*

$$\|u \cdot n\|_{H^{-1/2}(\partial O)} \le C\Big(\|\operatorname{div} u\|_{L^2(O)} + \|u\|_{L^2(O)}\Big),$$

which is weaker than the estimate stated in Lemma 2 and not good enough for the proof of Proposition 3. Lemma 2 is also useful in other situations see e.g. [7] and [16].

The third lemma is a consequence of Fredholm's theorem (see e.g., [4, 12]) and the fact that the system (1.15) has only zero solution.

Lemma 3. *Assume that* (1.15) *has only zero solution. Let* $g \in H^{-1/2}(\partial\Omega_2)$ *and* $v \in H^1(\Omega \setminus \Omega_2)$ *be the unique solution to the system*

$$\begin{cases} \operatorname{div}(A\nabla v) + k^2\sigma v = 0 & \text{in } \Omega \setminus \bar{\Omega}_2\ , \\ v = 0 & \text{on } \partial\Omega\ , \\ A\nabla v \cdot \eta = 0 & \text{on } \partial\Omega_2\ . \end{cases}$$

We have

$$\|v\|_{H^1(\Omega\setminus\Omega_2)} \le C\|g\|_{H^{-1/2}(\partial\Omega_2)},$$

for some positive constant C *independent of* g.

We are ready to state the main result of this section which implies the existence and uniqueness of u_δ in Theorem 2 and provides an a priori estimate for it.

Proposition 3. *Let* $g \in H^{-1}(\Omega)$ *(the duality of* $H^1_0(\Omega)$*). There exists a unique solution* $v_\delta \in H^1_0(\Omega)$ *to the equation*

$$\operatorname{div}(\varepsilon_\delta A\nabla v_\delta) + k^2\varepsilon_0\Sigma v_\delta = g \text{ in } \Omega.$$

Moreover, we have

$$\|v_\delta\|_{H^1(\Omega)} \le C\Big(\frac{1}{\delta}\|g\|_{H^{-1}(\Omega)} + \|g\|_{L^2(\Omega_1)} + \|g\|_{L^2(\Omega_2\setminus\bar{\Omega}_1)}\Big), \tag{3.1}$$

for some positive constant C *independent of* g *and* δ*, as* δ *is small* [7].

Remark 14. *We note here that the conditions* $g \in H^{-1}(\Omega)$, $g \in L^2(\Omega_2 \setminus \bar{\Omega}_1)$*, and* $g \in L^2(\Omega_1)$ *do not imply* $g \in L^2(\Omega_2)$*. Here is an example. Let* $v \in H^1_0(\Omega)$ *such that* $v \in H^2(\Omega_1)$, $v \in H^2(\Omega_2\setminus\bar{\Omega}_1)$ *but* $v \notin H^2(\Omega_2)$*. Set* $g = \Delta v$*. Then* $g \in H^{-1}(\Omega)$, $g \in L^2(\Omega_2\setminus\bar{\Omega}_1)$*, and* $g \in L^2(\Omega_1)$*, but* $g \notin L^2(\Omega_2)$*. Hence one cannot write* $\|g\|_{L^2(\Omega_1)} + \|g\|_{L^2(\Omega_2\setminus\bar{\Omega}_1)}$ *as* $\|g\|_{L^2(\Omega_2)}$ *in* (3.1).

[7] See Remark 14 for comments on (3.1)

Proof. The existence of v_δ will follow from its uniqueness by Fredholm's theorem (see e.g. [4, 12]). We now establish the uniqueness by showing that $v_\delta = 0$ if $v_\delta \in H^1_0(\Omega)$ is a solution to the equation

$$\operatorname{div}(\varepsilon_\delta A \nabla v_\delta) + k^2 \varepsilon_0 \Sigma v_\delta = 0 \text{ in } \Omega.$$

Multiplying this equation by $\bar{v}_\delta$ and integrating the obtained expression on Ω, we have

$$\int_\Omega \varepsilon_\delta \langle A \nabla v_\delta, \nabla v_\delta \rangle - k^2 \varepsilon_0 \Sigma |v_\delta|^2 = 0.$$

This implies, by considering the imaginary part,

$$\int_{\Omega_2 \setminus \Omega_1} \langle A \nabla v_\delta, \nabla v_\delta \rangle = 0.$$

It follows that v_δ is constant in $\Omega_2 \setminus \Omega_1$. Thus $v_\delta = 0$ in $\Omega_2 \setminus \Omega_1$ since $0 = \operatorname{div}(\varepsilon_\delta A \nabla v_\delta) + k^2 \varepsilon_0 \Sigma v_\delta = k^2 \varepsilon_0 \Sigma v_\delta$ in $\Omega_2 \setminus \Omega_1$. As a consequence,

$$\operatorname{div}(A \nabla v_\delta) + k^2 \Sigma v_\delta = 0 \text{ in } \Omega.$$

This implies $v_\delta = 0$ in Ω since (1.15) has only zero solution in $H^1(\Omega)$. The proof of the existence and uniqueness of v_δ completes.

We now establish (3.1) by contradiction arguments. Assume that there exists $(g_\delta) \subset H^{-1}(\Omega)$ such that

$$\|v_\delta\|_{H^1(\Omega)} = 1 \text{ and } \frac{1}{\delta}\|g_\delta\|_{H^{-1}} + \|g_\delta\|_{L^2(\Omega_1)} + \|g_\delta\|_{L^2(\Omega_2 \setminus \bar{\Omega}_1)} \to 0, \tag{3.2}$$

as δ goes to 0. Here $v_\delta \in H^1_0(\Omega)$ is the unique solution to the equation

$$\operatorname{div}(\varepsilon_\delta A \nabla v_\delta) + k^2 \varepsilon_0 \Sigma v_\delta = g_\delta \text{ in } \Omega. \tag{3.3}$$

Multiplying this equation by $\bar{v}_\delta$ and integrating the obtained expression on Ω, we have

$$\int_\Omega \varepsilon_\delta \langle A \nabla v_\delta, \nabla v_\delta \rangle - k^2 \varepsilon_0 \Sigma |v_\delta|^2 = - \int_\Omega g_\delta \bar{v}_\delta.$$

Considering the imaginary part and using the fact that

$$\frac{1}{\delta}\Big| \int_\Omega g_\delta \bar{v}_\delta \Big| \le \frac{1}{\delta}\|g_\delta\|_{H^{-1}} \|v_\delta\|_{H^1(\Omega)} \to 0 \text{ as } \delta \to 0, \text{ by (3.2)},$$

we obtain, by (1.3),

$$\|\nabla v_\delta\|_{L^2(\Omega_2 \setminus \Omega_1)} \to 0 \text{ as } \delta \to 0. \tag{3.4}$$

Applying Lemma 2, we have

$$\Big\| A \nabla v_\delta \cdot \eta \Big|_{\text{int}} \Big\|^2_{H^{-1/2}(\partial \Omega_2)} + \Big\| A \nabla v_\delta \cdot \eta \Big|_{\text{ext}} \Big\|^2_{H^{-1/2}(\partial \Omega_1)}$$
$$\le C \|A \nabla v_\delta\|_{L^2(\Omega_2 \setminus \Omega_1)} \Big(\|A \nabla v_\delta\|_{L^2(\Omega_2 \setminus \Omega_1)} + \|\operatorname{div}(A \nabla v_\delta)\|_{L^2(\Omega_2 \setminus \Omega_1)} \Big).$$

Since $\operatorname{div}(A\nabla v_\delta) - \varepsilon_\delta^{-1}\varepsilon_0 k^2 \Sigma v_\delta = \varepsilon_\delta^{-1} g_\delta$ in $\Omega_2 \setminus \Omega_1$, it follows from (3.2), (3.4), and the boundedness of A and Σ that

$$\Big\| A\nabla v_\delta \cdot \eta\Big|_{\text{int}}\Big\|^2_{H^{-1/2}(\partial\Omega_2)} + \Big\| A\nabla v_\delta \cdot \eta\Big|_{\text{ext}}\Big\|^2_{H^{-1/2}(\partial\Omega_1)} \to 0 \text{ as } \delta \to 0. \tag{3.5}$$

This implies

$$\Big\| A\nabla v_\delta \cdot \eta\Big|_{\text{ext}}\Big\|_{H^{-1/2}(\partial\Omega_2)} \to 0 \text{ as } \delta \to 0,$$

since $\varepsilon_\delta A\nabla v_\delta\Big|_{\text{ext}} \cdot \eta = \varepsilon_\delta A\nabla v_\delta\Big|_{\text{int}} \cdot \eta$ on $\partial\Omega_2$. By Lemma 3, we have

$$\|v_\delta\|_{H^1(\Omega\setminus\bar{\Omega}_2)} \le C\Big\| A\nabla v_\delta \cdot \eta\Big|_{\text{ext}}\Big\|_{H^{-1/2}(\partial\Omega_2)} \to 0 \text{ as } \delta \to 0. \tag{3.6}$$

As a consequence,

$$\|v_\delta\|_{H^{1/2}(\partial\Omega_2)} \to 0 \text{ as } \delta \to 0. \tag{3.7}$$

Combining (3.4) and (3.7), and using Lemma 1, we have

$$\|v_\delta\|_{L^2(\Omega_2\setminus\Omega_1)} \to 0 \text{ as } \delta \to 0. \tag{3.8}$$

We deduce from (3.4) and (3.8) that

$$\|v_\delta\|_{H^1(\Omega_2\setminus\Omega_1)} \to 0 \text{ as } \delta \to 0. \tag{3.9}$$

This implies

$$\|v_\delta\|_{H^{1/2}(\partial\Omega_1)} \to 0 \text{ as } \delta \to 0. \tag{3.10}$$

By Lemma 1, it follows from (3.2) and (3.10) that

$$\|v_\delta\|_{L^2(\Omega_1)} \to 0. \tag{3.11}$$

Multiplying equation (3.3) by $\bar{v}_\delta$ and integrating the obtained expression on Ω_1, we have

$$-\int_{\partial\Omega_1} A\nabla v_\delta \cdot \eta\, \bar{v}_\delta + \int_{\Omega_1} \varepsilon_\delta \langle A\nabla v_\delta, \nabla v_\delta\rangle - \int_{\Omega_1} k^2\varepsilon_0 \Sigma |v_\delta|^2 = -\int_{\Omega_1} g_\delta \bar{v}_\delta. \tag{3.12}$$

On the other hand,

$$\int_{\partial\Omega_1} A\nabla v_\delta \cdot \eta\, \bar{v}_\delta \to 0 \text{ as } \delta \to 0, \text{ by (3.5) and (3.10);} \tag{3.13}$$

$$\int_{\Omega_1} \Sigma |v_\delta|^2 \to 0 \text{ as } \delta \to 0, \text{ by (3.11);} \tag{3.14}$$

and

$$\Big|\int_{\Omega_1} g_\delta \bar{v}_\delta\Big| \le \|g_\delta\|_{L^2(\Omega_1)}\|v_\delta\|_{L^2(\Omega_1)} \to 0 \text{ as } \delta \to 0, \text{ by (3.2).} \tag{3.15}$$

A combination of (3.12), (3.13), (3.14), and (3.15) yields

$$\int_{\Omega_1} |\nabla v_\delta|^2 \to 0. \tag{3.16}$$

From (3.11) and (3.16), we obtain

$$\|v_\delta\|_{H^1(\Omega_1)} \to 0 \text{ as } \delta \to 0. \tag{3.17}$$

A combination of (3.6), (3.9), and (3.17) yields

$$\|v_\delta\|_{H^1(\Omega)} \to 0 \text{ as } \delta \to 0 :$$

we have a contradiction. The proof of (3.1) completes. □

3.2 Proof of Theorem 2

We follow the strategy in the proof of Theorem 1.
Case 1: f is compatible with the system. The proof is derived from the following steps:
Step 1: Let $v \in H^1_0(\Omega)$ be a solution to the equation

$$\operatorname{div}(\varepsilon_0 A \nabla v) + k^2 \varepsilon_0 \Sigma v = \varepsilon_0 f.$$

We prove that $v = NL(k, f)$.
Step 2: Define $u_0 := NI(k, f)$. We prove that $u_0 \in H^1_0(\Omega)$ is a solution to the equation

$$\operatorname{div}(\varepsilon_0 A \nabla u_0) + k^2 \varepsilon_0 \Sigma u_0 = \varepsilon_0 f.$$

Step 3: Prove that $(u_\delta)_{0<\delta<\delta_0}$ is bounded in $H^1(\Omega)$ if δ_0 is small enough.
Step 4: Prove that (u_δ) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to u_0 as δ goes to 0.

We note immediately here that as a consequence of Steps 1 and 2, $NI(k, f)$ is the unique solution in $H^1_0(\Omega)$ to the equation (1.14). We will only establish the proof of Steps 1 and 3. The proof of Steps 2 and 4 is similar to the one of Steps 2 and 4 in Case 1 of the proof of Theorem 1. The details are left to the reader.

Step 1: Assume $v \in H^1_0(\Omega)$ is a solution to the equation

$$\operatorname{div}(\varepsilon_0 A \nabla v) + k^2 \varepsilon_0 \Sigma v = \varepsilon_0 f \text{ in } \Omega.$$

Set

$$v_1 = v \circ F^{-1} \text{ in } \Omega_4 \setminus \Omega_2 \tag{3.18}$$

and

$$\hat{\varepsilon}_0 = \begin{cases} 1 & \text{if } x \in \Omega_3 \setminus \Omega_2, \\ -1 & \text{if } x \in \Omega_4 \setminus \Omega_3. \end{cases} \tag{3.19}$$

It follows from Proposition 1 that $v_1 \in H^1(\Omega_3 \setminus \Omega_2) \cap H^1_{loc}(\Omega_4 \setminus \bar{\Omega}_2)$ and v_1 satisfies

$$\operatorname{div}(\hat{\varepsilon}_0 F_* A \nabla v_1) + k^2 \hat{\varepsilon}_0 F_* \Sigma v_1 = \hat{\varepsilon}_0 F_* f \text{ in } \Omega_4 \setminus \bar{\Omega}_2, \tag{3.20}$$

and

$$v_1 = v \text{ on } \partial\Omega_2 \quad \text{ and } \quad F_* A \nabla v_1 \cdot \eta = A \nabla v \cdot \eta \Big|_{\text{ext}} \text{ on } \partial\Omega_2.$$

In the last identity, we use the fact that $F_* A \nabla v_1 \cdot \eta = -A \nabla v \cdot \eta \Big|_{\text{int}}$ on $\partial\Omega_2$ by Proposition 1 and $A \nabla v \cdot \eta \Big|_{\text{ext}} = -A \nabla v \cdot \eta \Big|_{\text{int}}$ on $\partial\Omega_2$ by the transmission condition on $\partial\Omega_2$. Define

$$\mathbf{U} = v_1 - v \text{ in } \Omega_3 \setminus \Omega_2. \tag{3.21}$$

Since $F_* A = A$ and $F_* \Sigma = \Sigma$ in $\Omega_3 \setminus \Omega_2$, it follows that

$$\begin{cases} \operatorname{div}(A \nabla \mathbf{U}) + k^2 \Sigma \mathbf{U} = F_* f - f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ \mathbf{U} = 0 & \text{on } \partial\Omega_2, \\ A \nabla \mathbf{U} \cdot \eta = 0 & \text{on } \partial\Omega_2. \end{cases} \tag{3.22}$$

From the unique continuation principle, see e.g., [22] and [1], and the fact that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$, we have

$$\mathbf{U} = U \text{ in } \Omega_3 \setminus \Omega_2. \tag{3.23}$$

Define the function v_2 on Ω as follows

$$v_2(x) = \begin{cases} v_1 \circ G^{-1} & \text{if } x \in \Omega_3, \\ v & \text{if } x \in \Omega \setminus \Omega_3. \end{cases} \tag{3.24}$$

Using (1.1) and applying Proposition 1, we have

$$\operatorname{div}(\hat{A} \nabla v_2) + k^2 \hat{\Sigma} v_2 = \hat{f} \text{ in } \Omega \setminus \partial\Omega_3, \tag{3.25}$$

and, on $\partial\Omega_3$,

$$\begin{aligned} \hat{A} \nabla v_2 \cdot \eta \Big|_{\text{ext}} - \hat{A} \nabla v_2 \cdot \eta \Big|_{\text{int}} &= A \nabla v \cdot \eta \Big|_{\text{ext}} + F_* A \nabla v_1 \cdot \eta \Big|_{\text{ext}} \\ &= A \nabla v \cdot \eta \Big|_{\text{ext}} - F_* A \nabla v_1 \cdot \eta \Big|_{\text{int}} \quad \text{(by (3.20))} \\ &= A \nabla v \cdot \eta \Big|_{\text{ext}} - F_* A \nabla (v + \mathbf{U}) \cdot \eta \Big|_{\text{int}} \quad \text{(by (3.21))}. \end{aligned}$$

It follows from (3.23) that

$$\hat{A} \nabla v_2 \cdot \eta \Big|_{\text{ext}} - \hat{A} \nabla v_2 \cdot \eta \Big|_{\text{int}} = -F_* A \nabla U \cdot \eta \Big|_{\text{int}}. \tag{3.26}$$

Since $G(x) = x$ on $\partial\Omega$, we also obtain, on $\partial\Omega_3$,

$$v_2 \Big|_{\text{ext}} - v_2 \Big|_{\text{int}} = v \Big|_{\text{ext}} - v_1 \Big|_{\text{ext}} = v \Big|_{\text{ext}} - v_1 \Big|_{\text{int}} = v \Big|_{\text{ext}} - (\mathbf{U} + v) \Big|_{\text{int}} = -U \Big|_{\text{int}}. \tag{3.27}$$

A combination of (3.25), (3.26), (3.27), and (1.21) yields

$$v_2 = W \text{ in } \Omega. \tag{3.28}$$

This implies that v satisfies

$$\begin{cases} \operatorname{div}(A\nabla v) + k^2 \Sigma v = f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ v = W\Big|_{\text{ext}} & \text{on } \partial\Omega_3, \\ A\nabla v \cdot \eta\Big|_{\text{int}} = A\nabla W \cdot \eta\Big|_{\text{ext}} & \text{on } \partial\Omega_3. \end{cases}$$

Since in the $3d$ case $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$, applying the unique continuation principle, see e.g. [22] and [1], we have

$$v = V \text{ in } \Omega_3 \setminus \bar{\Omega}_2. \tag{3.29}$$

From (3.24), (3.21), (3.18), it follows that

$$v = \begin{cases} v_2 & \text{in } \Omega \setminus \Omega_3, \\ \mathbf{U} + v_1 & \text{in } \Omega_3 \setminus \Omega_2, \\ v_1 \circ F & \text{in } \Omega_2 \setminus \Omega_1, \\ v_2 \circ G \circ F & \text{in } \Omega_1. \end{cases}$$

We deduce from (3.23), (3.28) and (3.29) that

$$v = \begin{cases} W & \text{in } \Omega \setminus \Omega_3, \\ V & \text{in } \Omega_3 \setminus \Omega_2, \\ (V + U) \circ F & \text{in } \Omega_2 \setminus \Omega_1, \\ W \circ G \circ F & \text{in } \Omega_1, \end{cases}$$

Here in the third equality, we also used (3.21) and (3.23). Therefore,

$$v = NL(k, f) \text{ in } \Omega.$$

Step 3: Set

$$v_\delta = u_\delta - u_0 \text{ in } \Omega.$$

We have, in Ω,

$$\operatorname{div}(\varepsilon_\delta A\nabla v_\delta) + k^2 \varepsilon_0 \Sigma v_\delta = \operatorname{div}(\varepsilon_\delta A\nabla u_\delta) - \operatorname{div}(\varepsilon_\delta A\nabla u_0) + k^2 \varepsilon_0 \Sigma u_\delta - k^2 \varepsilon_0 \Sigma u_0.$$

Since $\operatorname{div}(\varepsilon_\delta A\nabla u_0) + k^2 \varepsilon_0 \Sigma u_0 = \varepsilon_0 f$ and $\operatorname{div}(\varepsilon_0 A\nabla u_0) + k^2 \varepsilon_0 \Sigma u_0 = \varepsilon_0 f$, it follows that

$$\operatorname{div}(\varepsilon_\delta \nabla v_\delta) + k^2 \varepsilon_0 \Sigma v_\delta = \operatorname{div}\left[(\varepsilon_0 - \varepsilon_\delta) A\nabla u_0\right] \text{ in } \Omega.$$

Applying Proposition 2, we obtain

$$\begin{aligned}\|\nabla v_\delta\|_{L^2(\Omega)} &\le C\Big(\|\nabla u_0\|_{L^2(\Omega)} + \|\operatorname{div}(A\nabla u_0)\|_{L^2(\Omega_2\setminus\bar\Omega_1)} + \|\operatorname{div}(A\nabla u_0)\|_{L^2(\Omega_1)}\Big)\\ &\le C(\|\nabla u_0\|_{L^2} + \|f\|_{L^2(\Omega)}),\end{aligned}$$

which yields, since $u_\delta = v_\delta + u_0$,

$$\|\nabla u_\delta\|_{L^2(\Omega)} \le C(\|\nabla u_0\|_{L^2} + \|f\|_{L^2(\Omega)}).$$

Since $u_\delta \in H^1_0(\Omega)$, by the Poincaré inequality, it follows that

$$\|u_\delta\|_{H^1(\Omega)} \le C(\|\nabla u_0\|_{L^2} + \|f\|_{L^2(\Omega)}).$$

The proof of Step 3 completes.

Case 2: f is not compatible with the system. The proof of this case is the same as the one of Case 2 in the proof of Theorem 1. The details are left to the reader. $\square$

Remark 15. *Our analysis (mainly in Steps 1 and 2) would make media whose some region contains negative index materials behave like a folded space as noted in [14].*

4 Applications

This section is devoted to some applications of Theorems 1 and 2 for super lenses, cloaking, and illusion optics. The use of negative index for super lenses, cloaking, and illusion optics are not new as mentioned Section 1.1. The goal of this section is to present a more general setting for these applications whose the rigorous analysis is firmly established in the previous sections, hence to illuminate the general mechanism behind these phenomena.

4.1 Applications for super lenses

The first immediate application of Theorems 1 and 2 is to create perfect lenses. More precisely, consider the settings in Theorems 1 and 2. Assume f is compatible and $f = 0$ in Ω_3. Since $u_0 = u \circ G \circ F$ on $\partial\Omega_1$, one can focus light from $\partial\Omega_1$ to $\partial\Omega_3$; more precisely one has a "perfect lens".

Here are concrete examples. Let $0 < r_1 < r_2 < r_3$, $m, l > 1$, $\Omega_1 = B_{r_1}$, $\Omega_2 = B_{r_2}$, $\Omega_3 = B_{r_3}$ with $r_3 = r_2^m / r_1^{m-1}$, $\Omega_4 = \mathbb{R}^d$, $F(x) = r_2^m x/|x|^m$ and $G(x) = r_3^l x/|x|^l$. We recall here that B_r denotes the ball centered at the origin of radius r. We have

$$G \circ F(x) = r_3^l \Big|\frac{r_2^m}{|x|^m} x\Big|^{-l} \frac{r_2^m}{|x|^m} x = \frac{r_3^l}{r_2^{m(l-1)}} |x|^{ml-m-l} x.$$

Choose $m, l > 1$ such that

$$ml = m + l.$$

We obtain

$$G \circ F(x) = \frac{r_3^l}{r_2^{m(l-1)}} x = \frac{r_2^m}{r_1^m} x. \tag{4.1}$$

It follows that if f is compatible then

$$u_0(x) = u_0\Big(\frac{r_2^m}{r_1^m} x\Big) \text{ on } \partial B_{r_1} \text{ or } u_0(x) = u_0\Big(\frac{r_1^m}{r_2^m} x\Big) \text{ on } \partial B_{r_3}.$$

We have a perfect lens. We also have, from (4.1) and (1.7),

$$G_* F_* A(x) = \Big(\frac{r_2^m}{r_1^m}\Big)^{2-d} A\big(r_1^m x / r_2^m\big) \text{ in } B_{r_3}, \tag{4.2}$$

and

$$G_* F_* \Sigma(x) = \Big(\frac{r_2^m}{r_1^m}\Big)^{-d} \Sigma\big(r_1^m x / r_2^m\big) \text{ in } B_{r_3}.$$

Consider the case $d = 3$, $k = 0$, $A = I$ in $\Omega \setminus B_{r_2}$,

$$A = \frac{r_1^m}{r_2^m} I \text{ in } B_{r_1} \setminus B_{r_1^m / r_2^{m-1}}, \quad \text{ and } \quad A = F_*^{-1} A \text{ in } B_{r_2} \setminus B_{r_1}.$$

We have

$$A = I = F_* A = G_* F_* A \text{ in } B_{r_3} \setminus B_{r_2}.$$

Assume $f = 0$ for $r_1^m / r_2^{m-1} < |x| < r_3$ (this implies $V = W$). Then f is compatible and

$$u_0(x) = u_0\Big(\frac{r_2^m}{r_1^m} x\Big) \text{ in } B_{r_1}.$$

In particular,

$$u_0(x) = u_0\Big(\frac{r_2^m}{r_1^m} x\Big) \text{ in } B_{r_1^m / r_2^{m-1}}.$$

Pendry in [21], and Anantha Ramakrishna and Pendry in [24] (see also [23]) considered the case $m = l = 2$ and f is a dipole. We show here how to obtain similar phenomena for a class of transformations F and G.

Consider the case $d = 2$, $k = 0$, and $A = I$ in $\Omega \setminus \Omega_2$, $A = F_*^{-1} A$ in $B_{r_2} \setminus B_{r_1}$, and A is constant in Ω_1. From (4.2), we have

$$G_* F_* A = A \text{ in } B_{r_3}.$$

Thus the cylindrical B_{r_1} would to an outside observer appear magnified by a factor $r_3 = r_2^m / r_1^{m-1}$. This phenomena was first observed by Nicorovici-McPhedran-Milton [17] in the case $A = I$ in $\Omega \setminus \Omega_1$, $A = cI$ in Ω_1 for some constant $c > 0$, and $m = l = 2$.

4.2 Applications for cloaking and illusion optics

In this section, we discuss the possibility to obtain illusion optics and cloaking using negative index materials. Given A, Σ and A', Σ' respectively in $\Omega_3 \setminus \Omega_2$ and Ω_3, we choose A in Ω_2 such that $A = F_*^{-1} A$ and $\Sigma = F_*^{-1}\Sigma$ in $\Omega_2 \setminus \Omega_1$ and $A = F_*^{-1} G_*^{-1} A'$ and $\Sigma = F_*^{-1} G_*^{-1} \Sigma'$ in Ω_1. By Theorems 1 and 2, for observers outside Ω_3, Ω_3 would be characterized by the medium A', Σ' since $\hat{A}, \hat{\Sigma} = A', \Sigma'$ in Ω_3. Roughly speaking, A, Σ would appear like A', Σ': one obtains illusion optics. By choosing $A', \Sigma' = I, 1$ in Ω_3: one cloaks the region $\Omega_3 \setminus \Omega_2$. Here we assume that sources are compatible. The mechanism of illusion optics and cloaking described here was respectively proposed in [11] and [10] in the case Ω_1, Ω_2, and Ω_3 F, G, are radial symmetry. These mechanisms have supported by simulations and experiments. As far as we know, our paper provides the first rigorous analysis for the mechanism of cloaking and illusion optics given in these two papers. Nevertheless, we note that in [10], the authors only assumed that $A = F_*^{-1} A$ and $\Sigma = F_*^{-1}\Sigma$ in $\Omega_2 \setminus \Omega_1$; we do not know how to explain this situation using our analysis. The first observation on the cloaking using negative index materials (for the $2d$-quasi-static case) went back to [17]. Another interesting phenomena related to the mechanism of cloaking, we mentioned here, is the cloaking effects associated with anomalous localized resonance discussed in a very interesting paper of Milton and Nicorovici [13] (see also [15], [18] [5], [3] for related works). They showed that in the $2d$ quasi static regime, with $\Omega_1 = B_{r_1}$, $\Omega_2 = B_{r_2}$, and $A = I$, for a dipole source inside $B_{r_2^{3/2}/r_1^2} \setminus B_{r_2}$, $\|u_\delta\|_{H^1(\Omega)}$ goes to infinity; however u_δ converges outside $B_{r_2^{3/2}/r_1^2}$. This result has been recently extended in [2] where they characterized f such that this phenomena happens for arbitrary Ω_1 and Ω_2. In this paper, we provide a general class of f (f is compatible with the system) such that this phenomena does not take place for a general setting. It would be nice to characterize f so that one obtains the cloaking effects associated with anomalous localized resonance i.e. $\|u_\delta\|_{H^1(\Omega)}$ goes to ∞ while as u_δ converges outside a large region. We emphasize here that there is a difference in our characterization and the characterization given in [2]. In this paper, we characterize sources for which the total energy of the fields remains bounded as the loss parameter goes to zero. In [2], the authors characterize sources for which the dissipation energy blows up and the fields remains bounded outside a bounded region as the loss parameter goes to zero.

A Appendix: Scattered case

In this appendix, we present the corresponding results for scattered waves. For the convenience of the reader, we will recall the notation presented in Section 1.1. Let Ω_1, Ω_2, Ω_3, and Ω_4 be smooth subsets of $\mathbb{R}^d$ ($d = 2, 3$) such that they are connected,

$$\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega_4.$$

We assume that there exist $x_1 \in \Omega_1$ and $x_2 \in \Omega_3$, and two diffeomorphisms

$$F : \bar{\Omega}_2 \setminus \{x_1\} \to \Omega_4 \setminus \Omega_2, \quad G : \Omega_4 \setminus \Omega_3 \to \bar{\Omega}_3 \setminus \{x_2\}$$

such that there exist diffeomorphism extension of F and G on a neighborhood of $\partial\Omega_2$ and $\partial\Omega_3$ respectively,

$$\left| \begin{array}{c} F(\Omega_2 \setminus \Omega_1) = \Omega_3 \setminus \bar{\Omega}_2, \\ F(x) = x \text{ on } \partial\Omega_2 \quad \text{ and } \quad G(x) = x \text{ on } \partial\Omega_3, \end{array} \right.$$

and

$$G \circ F : \Omega_1 \to \Omega_3 \text{ is a diffeomorphism if one sets } G \circ F(x_1) = x_2.$$

The geometry is illuminated in Figure 1.

For $\delta \geq 0$, ε_δ is given as follows.

$$\varepsilon_\delta(x) := \begin{cases} -1 + i\delta & \text{if } \Omega_2 \setminus \Omega_1 \\ 1 & \text{otherwise }, \end{cases}$$

We will assume that A and Σ satisfy (1.3), (1.4),

$$A(x) = I \text{ and } \Sigma(x) = 1 \quad \text{ if } |x| \text{ is large enough},$$

and in the $3d$ case, we assume in addition that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$.

In this appendix, we investigate the behavior of $u_\delta \in H^1_{loc}(\mathbb{R}^d)$, as δ goes to 0, the solution to the equation

$$\operatorname{div}(\varepsilon_\delta A \nabla u_\delta) + k^2 \varepsilon_0 \Sigma u_\delta = \varepsilon_0 f \text{ in } \mathbb{R}^d, \tag{A1}$$

which satisfies the outgoing condition:

$$\partial_r u_\delta - iku_\delta = o(r^{(1-d)/2}) \text{ as } r = |x| \to \infty.$$

Definition A1. *We say that $f \in L^2(\mathbb{R}^d)$ ($d = 2$ or 3) with compact support is compatible with the system if there exist $U \in H^1(\Omega_3 \setminus \Omega_2)$ and $V \in H^1(\Omega_3 \setminus \Omega_2)$ such that*

$$\begin{cases} \operatorname{div}(A\nabla U) + k^2 \Sigma U = F_* f - f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ U = 0 & \text{on } \partial\Omega_2, \\ A\nabla U \cdot \eta = 0 & \text{on } \partial\Omega_2, \end{cases}$$

and

$$\begin{cases} \operatorname{div}(A\nabla V) + k^2 \Sigma V = f & \text{in } \Omega_3 \setminus \bar{\Omega}_2, \\ V = W\Big|_{\text{ext}} & \text{on } \partial\Omega_3, \\ A\nabla V \cdot \eta = A\nabla W \cdot \eta\Big|_{\text{ext}} & \text{on } \partial\Omega_3. \end{cases}$$

Here $W \in \cap_{R>0} H^1(B_R \setminus \partial\Omega_3)$ is the unique outgoing solution to the system

$$\begin{cases} \operatorname{div}(\hat{A}\nabla W) + k^2 \hat{\Sigma} W = \hat{f} & \text{in } \mathbb{R}^d \setminus \partial\Omega_3 \ , \\ [W] = -U & \text{on } \partial\Omega_3 \ , \\ [\hat{A}\nabla W \cdot \eta] = -A\nabla U \cdot \eta & \text{on } \partial\Omega_3 \ , \end{cases}$$

If f is compatible with the system, then $NI_s(k, f)$ is defined as follows

$$NI_s(k,f) = \begin{cases} W & \textit{if } x \in \mathbb{R}^d \setminus \Omega_3, \\ V & \textit{if } x \in \Omega_3 \setminus \Omega_2, \\ (U+V)\circ F & \textit{if } x \in \Omega_2 \setminus \Omega_1, \\ W \circ G \circ F & \textit{if } x \in \Omega_1. \end{cases} \tag{A2}$$

We are ready to state

Theorem A1. *Let $\delta > 0$, $d = 2$, or 3, $f \in L^2(\mathbb{R}^d)$ with compact support, and $u_\delta \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to the equation* (A1)*. Assume that $F_*A = A$ and $F_*\Sigma = \Sigma$ in $\Omega_3 \setminus \Omega_2$, and in the 3d case, assume in addition that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$. We have*

a) Case 1: f is compatible with the system. The sequence (u_δ) converges weakly in $H^1_{loc}(\mathbb{R}^d)$ and strongly in $L^2_{loc}(\mathbb{R}^d)$ to $u_0 \in H^1_{loc}(\mathbb{R}^d)$, the unique outgoing solution to the equation

$$\operatorname{div}(\varepsilon_0 \nabla u_0) + k^2 \varepsilon_0 \Sigma u_0 = \varepsilon_0 f \textit{ in } \mathbb{R}^d.$$

as $\delta \to 0$. Moreover, $u_0 = NL_s(k, f)$ where $NL_s(k, f)$ is given in (A2)*.*

b) Case 2: f is not compatible with the system. We have

$$\lim_{\delta \to 0} \|u_\delta\|_{H^1(B_r)} = +\infty \textit{ if } \Omega_3, \operatorname{supp} f \subset\subset B_r.$$

Assume in addition that $f = 0$ in Ω_3 and $G_*F_*A = A$ in $\Omega_3 \setminus \Omega_2$. Then $U = 0$ and $V = W$ in $\Omega_3 \setminus \Omega_2$, and $W \in H^1_{loc}(\mathbb{R}^d)$ is the outgoing solution to the equation

$$\operatorname{div}(\hat{A}\nabla W) + k^2 \Sigma W = f \text{ in } \mathbb{R}^d.$$

We have the following

Corollary A3. *Assume $G_*F_*A = F_*A = A$ in $\Omega_3 \setminus \Omega_2$, $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$, and $f = 0$ in Ω_3. Then f is compatible and (u_δ), the solutions to the equations* (A1)*, converges weakly in $H^1_{loc}(\mathbb{R}^d)$ and strongly in $L^2_{loc}(\mathbb{R}^d)$ to u_0 the unique outgoing solution in $H^1_{loc}(\mathbb{R}^d)$ to the equation*

$$\operatorname{div}(\varepsilon_0 A \nabla u_0) + k^2 \varepsilon_0 \Sigma u_0 = \varepsilon_0 f \textit{ in } \mathbb{R}^d,$$

as $\delta \to 0$. Moreover

$$u_0 = W \textit{ in } \mathbb{R}^d \setminus \Omega_3,$$

where $W \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to the equation

$$\operatorname{div}(\hat{A}\nabla W) + k^2 \hat{\Sigma} W = f \textit{ in } \mathbb{R}^d.$$

A cousin of Theorem 3 in which the loss parameter involves in the two terms is as follows:

Theorem A2. *Let $\delta > 0$, $d = 2$, or 3, $f \in L^2(\mathbb{R}^d)$ with compact support, and $u_\delta \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to the equation*

$$\operatorname{div}(\varepsilon_\delta A \nabla u_\delta) + k^2 \varepsilon_\delta \Sigma u_\delta = f \text{ in } \mathbb{R}^d.$$

Assume that $F_*A = A$ and $F_*\Sigma = \Sigma$ in $\Omega_3 \setminus \Omega_2$, and in the 3d case, assume in addition that $A \in C^1(\bar{\Omega}_3 \setminus \Omega_2)$. We have

a) Case 1: f is compatible with the system. The sequence (u_δ) converges weakly in $H^1_{loc}(\mathbb{R}^d)$ and strongly in $L^2_{loc}(\mathbb{R}^d)$ to $u_0 \in H^1_{loc}(\mathbb{R}^d)$, the unique outgoing solution to the equation

$$\operatorname{div}(\varepsilon_0 \nabla u_0) + k^2 \varepsilon_0 \Sigma u_0 = \varepsilon_0 f \text{ in } \mathbb{R}^d.$$

as $\delta \to 0$. Moreover, $u_0 = NL_s(k, f)$ where $NL_s(k, f)$ is given in (A2).

b) Case 2: f is not compatible with the system. We have

$$\lim_{\delta \to 0} \|u_\delta\|_{H^1(B_r)} = +\infty \textit{ if } \Omega_3, \operatorname{supp} f \subset\subset B_r.$$

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